10. Let A be a commutative Banach algebra and let Δ be the set of all complex homomorphisms of A. Prove the following.

(a) Every maximal ideal of A is the kernel of some $h \in \Delta$.

(b) If $h \in \Delta$, the kernel of h is a maximal ideal of A.

(c) An element $x \in A$ is invertible in A if and only if $h(x) \neq 0$ for every $h \in \Delta$.

(d) An element $x \in A$ is invertible in A if and only if x lies in no proper ideal of A.

(e) $\lambda \in \sigma(x)$ if and only if $h(x) = \lambda$ for some $h \in \Delta$.

2828/M11 **OCTOBER 2011**

Paper I — L^p SPACES AND BANACH ALGEBRA

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) If φ is convex on (a,b), then prove that φ is continuous on (a,b).

(b) State and prove Jensen's Inequality.

2. (a) If P and q are conjugate exponents, $1 \le p \le \infty$, and if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then show that $fg \in L^1(\mu)$, and $\|f+g\|_l \le \|f\|_p + \|g\|_q$.

(b) If $1 \le p \le \infty$, and f,g $\in L^p(\mu)$. Show that $f+g \in L^p(\mu)$ and $\|f+g\|_p \le \|f\|_p + \|g\|_q$.

3. Define the term conjugate exponents. Establish the Holder's and Minkowski's inequalities.

4. (a) Let X be the class of all complex, measurable simple functions on X such that $\mu(\{x : S(x) \neq 0\}) < \infty$. If $1 \leq p \leq \infty$ show that S is dense in $L^p(\mu)$.

(b) Show that if X is a locally compact Hausdorff space $C_o(X)$ is the completion of $C_c(X)$ under supremum metric.

5. (a) Define the terms : Banach Algebra, Complex Homomorphism.

(b) If A is Banach algebra, $x \in A$, $\|x\| < 1$. Show that

(i) $e - x$ is invertible

(ii) $\|(e-x)^{-1} - e - x\| \leq \frac{\|x\|^2}{1-\|x\|}$.

(iii) $|\phi(n) < 1|$ for every complex homomorphism ϕ on A.

6. If ϕ is a linear functional on a Banach algebra A, such that $\phi(e) = 1$ and $\phi(x) \neq 0$ for every invertible $x \in A$, then prove that

$$\phi(xy) = \phi(x)\phi(y).$$

7. If A is Banach algebra and $x \in A$. Prove that

(a) the spectrum $\sigma(x)$ of x is compact and non empty and

(b) the spectral radius, $\rho(x)$ of x satisfies

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}.$$

8. (a) Define the term : Gelfand Transforms.

(b) Let Δ be the maximal ideal space of a commutative Banach algebra A. Show that

(i) Δ is a compact Hausdorff space.

(ii) The Gelfand transform is a homomorphism of A onto subalgebra $\hat{A}$ of $C(\Delta)$, whose kernel is rad A.

(iii) If $x \in A$ the range of $\hat{x}$ is the spectrum $\sigma(x)$ and $\|\hat{x}\|_\infty = \rho(x) \leq \|x\|$.

9. (a) State and prove Gelfand-Naimark theorem.

(b) If ψ : B→A is a homomorphism of a commutative Banach algebra B into a Semi simple commutative Banach algebra A show that ψ is continuous.

2829/M12 **OCTOBER 2011**

MODULUS AND GALOIS THEORY

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Let R be a commutative ring with 1 which is a simple ring. Prove that R is a field.

   (b) Prove that any field is a simple ring.

2. (a) State and prove first isomorphism theorem.

   (b) Show that P is a prime ideal of Z iff either $P = 0$ or $P = pZ$ for some prime p.

3. (a) Let R be a principal ideal domain. Prove that every $a \in R$ which is not a unit can be expressed as a product of irreducible elements.

   (b) Show that the ring $R = \left\{ \frac{m}{n} \middle/ m, n \in Z, n odd \right\}$ is a principal ideal domain.