ALGEBRA — I

(For those who joined in July 2003 and after)

Time : Three hours Maximum : 100 marks

SECTION A — ($4 \times 10 = 40$ marks)

Answer any FOUR questions

1. Let G be a finite group. Prove that $OG = O(z(G)) + \sum_{|G:N(a)|>1} |G : N(a)|$.

2. Let p and q be two distinct primes. Prove that any group of order pq is solvable.

3. Let R be a ring with unit element? Prove that R is isomorphic to a subring of the ring of endomorphisms of the abelian group R.

4. Let R be a commutative ring with unit element. Prove that $R[x]$ is an integral domain.

5. Let F be a field and $f(x) \in F[x]$. Prove that $\alpha \in F$ is a root of $f(x)$ if and only if $x - \alpha$ divides $f(x)$.

6. Let R be a b and F be the quotient field of R. Let $f(x) \in R[x]$ be irreducible in $R[x]$. Prove that $f(x)$ is also irreducible in $F[x]$.

7. Let K/F be a finite separable extension. Prove that $K = F(\alpha)$ for some $\alpha \in K$.

8. Let K/F be a Galoi's entension of degree n. Prove that $G(K/F)$ is a group of order n and F is the fined field of $G(K/F)$.

SECTION B — ($3 \times 20 = 60$ marks)

Answer any THREE questions.

9. Prove that G is a direct product of subgroups H if and only if

(a) Every $x \in G$ can be uniquely expressed as $x = hk$, $h \in H$, $k \in K$.

(b) $hk = kh$ $h \in H$, $k \in K$.

10. State and prove Cauchy's theorem.

11. State and prove Gauss theorem.

12. Let K/F be a finite extension. Prove that K/F is separable if and only if the number of F-isomorphisms of K in N, for any finite normal extension N/F, $K \subset N$ is equal to [K : F].

13. Let G be a finite group of automorphisms of K and the fined F of G, then prove that K/F is a Galoi's extension with $G(K/F) = G$.

14. State and prove the fundamental theorem of Galoi's theory.

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REAL AND COMPLEX ANALYSIS

(For those who joined in July 2003 and after)

Time : Three hours Maximum : 100 marks

PART A — (4 × 10 = 40 marks)

Answer any FOUR questions.

1. (a) If $x \in R$, $y \in R$ and $x > 0$, then prove that there is a positive integer n such that $nx > y$.

   (b) If $x \in R$, $y \in R$ and $x < y$, prove that there exists a rational number $p \in Q$ such that $x < p < y$.

2. State and prove the ratio test.

3. If f is a continuous mapping of a metric space X into a matric space Y and if E is a connected subset of X, prove that f(E) is connected.

4. State and prove the fundamental theorem of Calculus.

5. Derive Cauchy - Riemann equations.

6. State and prove the Liouville's theorem.

7. State and prove the argument principle.

8. State and prove the Hurwitz theorem.

PART B — (3 × 20 = 60 marks)

Answer any THREE questions.

9. For any sequence $\{c_n\}$ of positive numbers prove that $\lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n} \le \lim_{n\to\infty} \inf \sqrt[n]{c_n}$ and

$\lim_{n\to\infty} \sup \sqrt[n]{c_n} \le \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$.

10. State and prove the Taylor's theorem for a real function.

11. Prove that $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.

12. State and prove that Cauchy's theorem.

13. State and prove the Residue theorem.

14. State and prove that Weierstrass theorem.