OCTOBER 2011

ALGEBRA — I

(For those who joined in July 2002 or earlier)

Time : Three hours

Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) State and Prove Sylow's theorem for Abelian groups.

   (b) If G is a group, prove that

   (i) The identity element of G is unique.

   (ii) Every $a \in G$ has a unique inverse in G.

   (iii) For every $a \in G, \left(a^{-1}\right)^{-1} = a$.

   (iv) For all $a, b \in G, (a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$.

2. (a) State and Prove Cayley's theorem.

   (b) If ϕ is a homomorphism of G into $\overline{G}$ with kernel K, prove that K is a normal subgroup of G.

3. (a) State and Prove Cauchy's Theorem for Abelian Groups.

(b) Prove that every finite Abelian group is the direct product of cyclic groups.

4. (a) Prove that every integral domain can be imbedded in a field.

(b) Let R be a commutative ring with unit element whose only ideas are (0) and R itself, prove that R is a field.

5. (a) State and prove The Division Algorithm.

(b) Prove that the ideal $A=(p(x))$ in $F[x]$ is a maximal ideal iff $p(x)$ is irreducible over F.

6. (a) If V is a finite-dimensional vector space and if $u_1,\ldots\ldots,u_m$ span V, prove that some subset of $u_1,\ldots\ldots,u_m$ forms a basis of V.

(b) If V is finite-dimensional and if W is a subspace of V, prove that W is finite-dimensional, $\dim W \leq \dim V$ and $\dim V/W = \dim V - \dim W$.

7. (a) If V and W are of dimensions m and n respectively over F, prove that Hom (V, W) is of dimension mn over F.

(b) Let V be a finite-dimensional inner product space, prove that V has an Orthonormal set as a basis.

8. (a) If $T \in A(V)$ has all its characteristic roots in F, prove that there is a basis of V in which the matrix of T is triangular.

(b) If M of dimension m, is cyclic with respect to T, prove that the dimension of MT^k is $m-k$ for all $k \leq m$.

9. (a) Prove that two nilpotent linear transformations are similar iff they have the same invariants.

(b) Prove that the elements S and T in $A_F(V)$ are similar in $A_F(V)$ iff they have the same elementary divisors.

10. (a) Prove that $T \geq 0$ iff $T = AA^*$ for some A.

(b) Prove that A is invertible iff det $A \neq 0$.

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9. State and prove that Stone-Weierstrass theorem.

10. (a) State and prove Parseval's theorem.

(b) If f is a positive function on $(0, \infty)$ such that

(i) $f(x+1) = xf(x)$

(ii) $f(1) = 1$

(iii) $\log f$ is convex

then prove that $f(x) = \Gamma(x)$.

2517/KBO

OCTOBER 2011

Paper II — ANALYSIS — I

(For those who joined in July 2002 or earlier)

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) If a and b are positive real numbers and n is a positive integer, then prove that $(ab)^{1/n} = a^{1/n} b^{1/n}$.

(b) For any real $x > 0$ and every integer $n > 0$ prove that there is one and only one y such that $y^n = x$.

(c) Prove that every infinite subset of a countable set A is countable.

2. (a) Let X be a metric space. $E \subset X$, If p is a limit point of a set E, then prove that every neighborhood of p contains infinitely many points.