8728/M11 **APRIL 2009**

Paper I — L_p SPACES AND BANACH ALGEBRAS

(Held in November 2009)

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Prove that a real differentiable function ϕ is convex in (a, b) if and only if $a < s < t < b \Rightarrow \phi'(s) < \phi'(t)$.

   (b) If ϕ is convex on (a, b) prove that ϕ is continuous on (a, b).

2. Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space with measure μ. Let f and g be measurable functions on X with range in $[0, \infty)$. Prove that

   (a) $\int_X fg\,d\mu \leq \left[\int_X f^p\,d\mu\right]^{1/p}\left[\int_X g^q\,d\mu\right]^{1/q}$

   (b) $\left[\int_X (f+g)^p\,d\mu\right]^{1/p} \leq \left[\int_X f^p\,d\mu\right]^{1/p} + \left[\int_X g^q\,d\mu\right]^{1/q}$

3. Prove that $L^p(\mu)$ is a complete metric space for $1 < p < \infty$ and for every positive measure μ.

4. State and prove Lusins theorem.

5. If X is a locally compact Hausdorff space, prove that the completion of $C_c(X)$ relative to the metric defined by the supremum norm $\|f\| = \sup_{x \in X} |f(x)|$.

6. Let ϕ be a linear functional on a Banach algebra A such that $\phi(e) = 1$ and $\phi(x) = 0$ for every invertible $x \in A$. Prove that $\phi(xy) = \phi(x)\phi(y)$ for every $x, y \in A$.

7. (a) Let A be a Banach algebra and let $x \in A$. Prove that the spectrum $\sigma(x)$ is a compact nonempty subset of C.

   (b) Let A be a Banach algebra and let $x \in A$. Prove that $P(x) = \lim_{n \to \infty} \|x^n\|^{1/n} = \inf_{n \geq 1} \|x^n\|^{1/n}$.

8. (a) State and prove Gelfand Mazur theorem.

   (b) Let A be a commutative Banach algebra. Prove that :

   The Gelfand transform $X \to \hat{X}$ from $A \to \hat{A}$ preserves the norm (isometry) if and only if $\|x^2\| = \|x\|^2$ for every $x \in A$.

9. If A is a Banach algebra with an involution, and if $x \in A$, then prove the following :

   (a) $x + x^*$, $i(x - x^*)$ and xx^* are hermitian.

   (b) x has a unique representation $x = u + iv$ where $u, v \in A$ and u and v are hermitian.

   (c) the unit element e is hermitian.

   (d) x is invertible in A if and only if x^* is invertible in which case $(x^*)^{-1} = (x^{-1})^*$.

   (e) $\lambda \in \sigma(x)$ iff $\bar{\lambda} \in \sigma(x^*)$.

10. (a) If A is a commutative B^* algebra which contains an element x such that the polynomials in x and x^* are dense in A. Prove that the formula $(\psi f)\hat{} = f \circ \hat{x}$ defines an isometric isomorphism ψ of $C(\sigma(x))$ onto A which satisfies $\psi \hat{f} = (\psi f)^*$ for every $f \in C(\sigma(x))$. Prove also that if $f(\lambda) = \lambda$ on $\sigma(x)$, then $\psi f = x$.

    (b) Suppose A is a commutative Banach algebra with an involution. If x is a self adjoint element of A and if $\sigma(x)$ contains no real number $\lambda \leq 0$, prove that there exists $y \in A$ with $y^2 = x$ and $y = y^*$

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8. (a) State and prove the inversion theorem.

(b) Define a Banach algebra. Show that L' is a Banach algebra if we define multiplication by convolution.

9. State and prove the Plancheral theorem.

10. State and prove the theorem due to Paley and Wiener.

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8730/M21 **APRIL 2009**

Paper III — FOURIER TRANSFORMS
(Held in November 2009)

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Define outer measure of a subset A of R. Prove that if $\{A_n\}$ is a countable collection of subsets of R, then $m^*(UA_n) \leq \sum m^*(A_n)$.

(b) Prove that the collection m of all measurable sets in R is a σ-algebra.

2. (a) Prove that every Borel set is measurable.

(b) Construct a nonmeasurable subset of R.

3. (a) If f and g are measurable sets, prove that $f+c$, cf, $f+g$, $f-g$ and fg are measurable sets for any constant c.